Nontrivial edge coupling from a Dirichlet network squeezing: the case of a bent waveguide

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Abstract

In distinction to the Neumann case the squeezing limit of a Dirichlet network leads in the threshold region generically to a quantum graph with disconnected edges, exceptions may come from threshold resonances. Our main point in this paper is to show that modifying locally the geometry we can achieve in the limit a nontrivial coupling between the edges including, in particular, the class of δ-type boundary conditions. We work out an illustration of this claim in the simplest case when a bent waveguide is squeezed.

1 Introduction

Quantum mechanics on graphs attracted a lot of attention recently – let us just mention the proceeding volume [BCFK06] as a guide to the abundant bibliography in the field. The interest has different sources, important among them are numerous existing and potential applications. While simple and versatile, however, quantum graph models have a problem: in a sense they offer too much freedom. The requirement of self-adjointness, or probability current conservation, determines a class of boundary conditions which connect the column vectors Ψ and Ψ' of boundary values of the wave functions and their derivatives at a given graph vertex. Following [Ha00, KS00] these conditions can be cast into the unique form

$$(U-I)\Psi + i(U+I)\Psi' = 0\,, \tag{1.1}$$

where U is an $n\times n$ unitary matrix, n being the number of edges sprouting of the vertex.

Hence a vertex coupling contains n^2 free parameters to be fixed. Asking about the meaning of various couplings one can naturally get useful insights by obtaining boundary conditions through limit of families of (regular or singular) interactions on the graph [E96, CE04, ET07].

A proper understanding of the problem requires, however, to find an interpretation of the coupling in terms of models without free parameters; a natural idea is to investigate the motion of a free quantum particle on a system of thin tubes which shrink towards the graph.

This is a longstanding and nontrivial problem and the answer depends substantially on the network dynamics we work with. In the case when the Hamiltonian (which can be identified with the Laplacian by a suitable choice of units) refers to tubes with Neumann boundary, the limit yields typically quantum graphs with free boundary conditions (often called not quite appropriately Kirchhoff) in the vertices, described by $U = \frac{2}{n}\mathcal{J} - I$, where $\mathcal{J}$ is the $n \times n$ matrix whose all entries are equal to one – see [FW93, KZ01, RS01, Sa01], the same is true also for shrinking families of "sleeve" manifolds without a boundary [EP05]; the convergence is norm-resolvent [Po05a] and the conclusion extends to resonances on such structures [EP07].

From the viewpoint of application to semiconductor structures and similar objects, however, it is the case of Dirichlet (hard-wall) boundary which is more important. It is very different from its Neumann counterpart and more difficult and relevant results started appearing only recently. The main source of the difficulties is that in the Dirichlet case geometric perturbations like bending, "swelling", twisting or branching give rise to an effective interaction – see, e.g., [EŠ89, DE95, EEK05] and references therein – sometimes attractive, sometimes repulsive, which changes spectral and scattering properties of such networks. Even the statement of the problem is more complicated than in the Neumann case where we naturally investigate spectrum around the zero value which is the continuum threshold. The analogous quantity in a Dirichlet network blows up to infinity and one has to choose the reference point by a suitable energy renormalization. In most cases, with the notable exception of the recent paper [MV06], the attention is paid to the vicinity of the "running" threshold.

It is generally conjectured, that the generic limit in the vicinity of the threshold corresponds to a fully decoupled graph with Dirichlet conditions at the edge endpoints. This is obvious if the vertex regions squeeze faster than tubes as in [Po05b] but it is expected to hold even without such an additional effective repulsion. To see the reason one has to realize that the problem at hand can be by scaling rephrased as analysis of a network of tubes of constant cross section whose overall size grow. This means that the distances between the "vertices" tend to infinity and the character of solutions to the Schrödinger equation is determined by its asymptotic properties around each single vertex. Away of the threshold, as in the particular case of [MV06] cited above, the limiting boundary conditions are generally nontrivial and given by the scattering properties of the "fat star" region. Around the threshold, on the other hand, the scattering is generically suppressed in view of the mentioned effective interaction – see a brief discussion in [MV06], the forthcoming paper [Gr07] and also a related problem with Dirichlet boundary replaced by a confining potential in [DT06] – leading to the Dirichlet decoupling.

This limiting behaviour is not universal, though. The situation changes if the described system associated with the vertex possesses a threshold resonance. The simplest case where a nontrivial effect of this type can be observed is a bent tube which squeezes in the limit to a graph of two halfline edges joined in a single vertex [ACF07]: in presence of a threshold

resonance one gets the line with a point interaction of scale-invariant type – cf. [HC06] and references therein. More general results of that type were announced in [Gr07].

Our main point in this paper is to argue than one can go further and construct classes of squeezing Dirichlet networks which produce wider families of vertex coupling when renormalized to the continuum threshold, including those with nonempty discrete spectrum or resonances. The procedure we propose consists of two steps:

(i) choose a network collapsing to a graph in such a way that the limit Hamiltonian has a threshold resonance[1]

(ii) change the scaling properties of the vertex region slightly, typically by adding higher order terms in the scaling parameter

The modification in point (ii) can be achieved in various ways, for instance, one can "wiggle" the edges angles or scale the vertex volume region at a rate which differs from that of the "edge tubes" by a higher order term, a combination of such perturbations, etc. Incidentally, the same effect can also be obtained by introducing suitable potentials into the vertex region, but a purely geometric way is probably the most interesting.

Notice that the described approximation follows the same scheme which one uses when interpreting pseudopotentials, or point interactions in dimensions two and three, by a suitable nonlinear scaling starting from a threshold resonance [AGHH05]. On the other hand, there is a large difference between the two cases coming from the fact that the geometric approximation discussed here covers not a single operator class but a broad variety of systems. Consequently, the proposal made above has a status of *a conjecture* and the corresponding procedure must be made concrete and worked out properly in each particular case.

To show that this programme is not void we are going in the rest of the paper perform this task for the bent-waveguide system studied in [ACF07]. We will show that modifying the bending angle around a threshold resonance value we can arrive at the limit at a two-parameter class of point interactions on the line including the important particular case of the δ interaction. In our model the system has multiple threshold resonances and the limiting procedure described above can be associated with each of them, we expect that this is a standard behavior of networks with Dirichlet boundary. We will describe the model and state the results in the next section. Then we extend the analysis of short-range potentials from [ACF07] to more general scaling, and in the last section we will prove our main theorem.

[1] The original network Hamiltonian at that may or may not have such a resonance, depending on the limiting procedure used. A threshold resonance may be present, e.g., in the leading term of the perturbation expansion w.r.t. the squeezing parameter as in the model discussed below.

2 The bent-waveguide model and the results

As we have said we use the model studied in [ACF07], namely a planar waveguide of constant width obtained by "fattening" a fixed smooth curve along its normal. We denote by C a curve embedded in $\mathbb{R}^2$, i.e. $C := \{(x, y) \in \mathbb{R}^2 |\, x = \gamma_1(s),\, y = \gamma_2(s),\, s \in \mathbb{R}\}$, assuming that it is parameterized by its arc length, $\gamma_1'^2 + \gamma_2'^2 = 1$. Moreover, we denote by $\gamma(s)$ the signed curvature of C,

$$\gamma(s) := \gamma_2'(s)\gamma_1''(s) - \gamma_1'(s)\gamma_2''(s)\,;$$

it completely characterizes the curve C up to Euclidean transformations and the curvature radius at a given point is given by $r = |\gamma|^{-1}$. We suppose that the curve is not self-intersecting, i.e., it has no loops, and for simplicity we consider only curves with a compactly supported curvature. The last assumption means that the curve C is made up of two straight half lines joined by a smooth curve. In particular, the (overall) bending angle of C is the angle between the two half lines, which is equal to $\theta = \int_{\mathbb{R}} \gamma(s)\, ds$.

The above mentioned fat curve which is our waveguide is the open set $\Omega \in \mathbb{R}^2$ defined as

$$\Omega := \{(x, y) \in \mathbb{R}^2 |\, x = \gamma_1(s) - u\gamma_2'(s), y = \gamma_2(s) + u\gamma_1'(s), s \in \mathbb{R}, u \in (-d, d)\}\,,$$

where s and u represent a global system of coordinates in strip, s being the coordinate along the curve while u is the distance along the normal to C. The width of the waveguide is constant and equal to $2d$ where $d > 0$. Another standard assumption we made is that d is smaller than the curvature radius, $d\|\gamma\|_\infty < 1$. The closure of Ω is conventionally denoted by $\overline{\Omega}$. The (negative) Laplacian with Dirichlet boundary conditions on $\partial\overline{\Omega}$, denoted as $-\Delta_\Omega^D$, is the Friedrichs extension of the positive, symmetric operator $L_0 := -\Delta$ with $\mathscr{D}(L_0) := C_0^\infty(\Omega)$.

The main geometric object of our study will be a family of waveguides whose shape and width depend on a scaling parameter $0 < \varepsilon \leqslant 1$ according to

$$\gamma_\varepsilon(s) := \frac{\sqrt{\lambda(\varepsilon)}}{\varepsilon}\gamma\left(\frac{s}{\varepsilon}\right); \qquad d_\varepsilon := \varepsilon^\alpha d\,, \quad \text{with } \alpha > 1\,, \tag{2.1}$$

where $\lambda(\varepsilon)$ is a fixed function to be specified below; its presence is the main difference comparing to [ACF07] because of the term $\sqrt{\lambda(\varepsilon)}$. We suppose that $\lambda(\varepsilon)$ is real, positive and analytic near the origin, and moreover, that it expands as

$$\lambda(\varepsilon) = 1 + \lambda_1\varepsilon + \mathcal{O}(\varepsilon^2) \qquad \text{with} \quad \lambda_1 = \lambda'(0)\,. \tag{2.2}$$

We assume that the condition $d_\varepsilon\|\gamma_\varepsilon\|_\infty < 1$ is satisfied for every $0 < \varepsilon \leqslant \varepsilon_0$ with some $\varepsilon_0 > 0$. The scaling (2.1) gives rise to a family of curves, $C_\varepsilon := \{(x, y) \in \mathbb{R}^2 |\, x = \gamma_{\varepsilon,1}(s),\, y = \gamma_{\varepsilon,2}(s),\, s \in \mathbb{R}\}$, the bending angle θ_ε of which changes slightly with respect to ε,

$$\theta_\varepsilon = \int_{\mathbb{R}} \gamma_\varepsilon(s)\, ds = \theta\sqrt{\lambda(\varepsilon)} = \theta\left(1 + \frac{1}{2}\lambda_1\varepsilon\right) + \mathcal{O}(\varepsilon^2)\,.$$

They in turn generate a family of bent waveguides, i.e. domains Ω_ε defined by

$$\Omega_\varepsilon := \{(x,y) \in \mathbb{R}^2 |\, x = \gamma_{\varepsilon,1}(s) - u\gamma'_{\varepsilon,2}(s), y = \gamma_{\varepsilon,2}(s) + u\gamma'_{\varepsilon,1}(s), s \in \mathbb{R}, u \in (-d_\varepsilon, d_\varepsilon)\}\,.$$

In the limit $\varepsilon \to 0$ the strip family shrinks to a graph, denoted by $\mathcal{G}$, made up of two edges and one vertex. Our aim is to investigate the limit of the respective operator family $-\Delta^D_{\Omega_\varepsilon}$ when $\varepsilon \to 0$. We will show that it approximates in a suitable sense an operator on $\mathcal{G}$, namely the Schrödinger operator on the line with a point interaction depending on γ and $\lambda(\varepsilon)$.

Before to state our main theorem, let us introduce some notation and mention some preliminary facts. Writing for brevity $\Omega' = \mathbb{R}\times(-d,d)$, we recall the following result [DE95, EŠ89]:

Proposition 1. *For any $0 < \varepsilon \leqslant \varepsilon_0$ let C_ε be as described above, with γ piecewise C^2 and compactly supported, such that γ', γ'' are bounded. Then $-\Delta^D_{\Omega_\varepsilon}$ is unitarily equivalent to the operator H_ε defined as the closure of the e.s.a. operator $H_{0\varepsilon}$ acting on $L^2(\Omega')$ as*

$$H_{0\varepsilon} = -\frac{\partial}{\partial s}\frac{1}{(1+\varepsilon^{\alpha-1}u\sqrt{\lambda(\varepsilon)}\gamma(s/\varepsilon))^2}\frac{\partial}{\partial s} - \frac{1}{\varepsilon^{2\alpha}}\frac{\partial^2}{\partial u^2} + \frac{1}{\varepsilon^2}V_\varepsilon(s,u)\,,$$

with the effective potential

$$V_\varepsilon(s,u) = -\frac{\lambda(\varepsilon)\gamma(s/\varepsilon)^2}{4(1+\varepsilon^{\alpha-1}u\sqrt{\lambda(\varepsilon)}\gamma(s/\varepsilon))^2} + \frac{\varepsilon^{\alpha-1}u\sqrt{\lambda(\varepsilon)}\gamma''(s/\varepsilon)}{2(1+\varepsilon^{\alpha-1}u\sqrt{\lambda(\varepsilon)}\gamma(s/\varepsilon))^3} - \frac{5}{4}\frac{\varepsilon^{2\alpha-2}u^2\lambda(\varepsilon)\gamma'(s/\varepsilon)^2}{(1+\varepsilon^{\alpha-1}u\sqrt{\lambda(\varepsilon)}\gamma(s/\varepsilon))^4}$$

and $\mathscr{D}(H_{0,\varepsilon}) = \{\psi \in L^2(\Omega') |\, \psi \in C^\infty(\Omega')\,,\, \psi(s,d) = \psi(s,-d) = 0\,,\, H_{0\varepsilon}\psi \in L^2(\Omega')\}$.

Let us next introduce the transversal modes, i.e., the normalized functions $\phi_n(u)$ which solve the equation $-\varepsilon^{-2\alpha}\phi''_n(u) = E_{\varepsilon,n}\phi_n(u)$ with the boundary conditions $\phi_n(\varepsilon^\alpha d) = \phi_n(-\varepsilon^\alpha d) = 0$. In particular, the corresponding eigenvalues $E_{\varepsilon,n}$ are explicitly given by

$$E_{\varepsilon,n} = \left(\frac{n\pi}{2d\varepsilon^\alpha}\right)^2 \qquad \text{with} \quad n = 1,2,\dots\,.$$

The resolvent of H_ε admits an integral representation with the kernel $(H_\varepsilon - z)^{-1}(s,u,s',u')$ for every $z \in \rho(H_\varepsilon)$ with Im $\sqrt{z} > 0$, where $\rho(H_\varepsilon)$ is the resolvent set of H_ε, cf. Thm II.37 in [Si71]. Using it we define the projection of the resolvent on the normal modes eigenspaces as

$$\overline{R}^\varepsilon_{n,m}(k^2,s,s') := \int_{-d}^{d} du\, du'\, \phi_n(u)(H_\varepsilon - k^2 - E_{\varepsilon,m})^{-1}(s,u,s',u')\phi_m(u')\,.$$

The operators $\overline{R}^\varepsilon_{n,m}(k^2) : L^2(\mathbb{R}) \to \mathrm{Ran}\big[\,\overline{R}^\varepsilon_{n,m}(k^2)\big] \subset L^2(\mathbb{R})$ introduced in this way are bounded operator-valued analytic functions of k^2 for all $k^2 \in \mathbb{C}\backslash\mathbb{R}$ and Im $k > 0$.

Next we have to recall some facts about one-dimensional Schrödinger operators. We say that the Hamiltonian

$$\overline{H} = -\frac{d^2}{ds^2} + \overline{V}(s) \tag{2.3}$$

has a zero energy resonance if there exist a function $\psi_r \in L^\infty(\mathbb{R})$, $\psi_r \notin L^2(\mathbb{R})$, such that $\overline{H}\psi_r = 0$ holds in the sense of distributions. In particular, if

$$\int_{\mathbb{R}} \overline{V}(s)\, ds \neq 0 \quad \text{and} \quad e^{a|\cdot|}\overline{V} \in L^1(\mathbb{R}) \tag{2.4}$$

holds for some $a > 0$, then exactly one of the following situations can occur [BGW85]:

Case I: The Hamiltonian $\overline{H}$ does not have a zero energy resonance.

Case II: The Hamiltonian $\overline{H}$ has a zero energy resonance; in such a case the function ψ_r can be chosen real and two real constants can be defined,

$$c_1 = \left[\int_{\mathbb{R}} \overline{V}(s)ds\right]^{-1} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{V}(s)\frac{|s-s'|}{2}\overline{V}(s')\psi_r(s')\, ds\, ds' \,, \qquad c_2 = -\frac{1}{2}\int_{\mathbb{R}} s\,\overline{V}(s)\psi_r(s)\, ds\,, \tag{2.5}$$

and moreover, c_1 and c_2 cannot vanish simultaneously. Let us stress that the constants c_1 and c_2 defined in (2.5) coincide with those employed in [ACF07].

Let us next introduce a pair of Hamiltonians on $\mathcal{G}$ both acting as $f \mapsto -f''$ but differing by the boundary conditions in the vertex. The first is the Dirichlet-decoupled operator $\overline{H}^d$ with the domain $\mathscr{D}(\overline{H}^d) := \{f \in H^2(\mathbb{R}\setminus 0) \cap H^1(\mathbb{R})|\, f(0) = 0\}$. The other is a point-interaction Hamiltonian $\overline{H}^r$, which again acts as $\overline{H}^r f = -f''$ but on the domain

$$\mathscr{D}(\overline{H}^r) := \Big\{ f \in H^2(\mathbb{R}\setminus 0)|\, (c_1+c_2)f(0^+) = (c_1-c_2)f(0^-)\,,$$
$$(c_1-c_2)f'(0^+) = (c_1+c_2)f'(0^-) + \frac{\hat{\lambda}}{c_1+c_2}f(0^-)\Big\} \quad \text{for } c_2 \neq -c_1\,;$$
$$\mathscr{D}(\overline{H}^r) := \Big\{ f \in H^2(\mathbb{R}\setminus 0)|\, f(0^-) = 0\,,\; f'(0^+) = \frac{\hat{\lambda}}{4c_1^2}f(0^+)\Big\} \quad \text{for } c_2 = -c_1\,,$$

where we put

$$\hat{\lambda} := \lambda_1 \int_{\mathbb{R}} \overline{V}(s)\big(\psi_r(s)\big)^2\, ds\,. \tag{2.6}$$

The graph $\mathcal{G}$ identifies naturally with a line and both $\overline{H}^d$ and $\overline{H}^r$ belong to the family of self-adjoint extensions of the symmetric operator $\overline{L}_0 f := -f''$ with the domain $\mathscr{D}(\overline{L}_0) := C_0^\infty(\mathbb{R}\backslash\{0\})$ [ABD95].

Let us say a few more words on the family $\overline{H}^r$ which obviously depends on two real parameters. It is a straightforward exercise to check that the boundary conditions appearing

in the definition of $\mathscr{D}(\overline{H}^r)$ can be rewritten in the form (1.1) with $\Psi := (f(0^+), f(0^-))^T$, $\Psi' := (f'(0^+), -f'(0^-))^T$ and the 2×2 unitary matrix

$$U := \frac{1}{2(c_1^2+c_2^2)+i\hat\lambda}\begin{pmatrix} -4c_1c_2 - i\hat\lambda & 2(c_1^2-c_2^2) \\ 2(c_1^2-c_2^2) & 4c_1c_2 - i\hat\lambda \end{pmatrix}. \tag{2.7}$$

In particular, for $\lambda_1 = 0$ the boundary conditions define the "scale invariant" Hamiltonian obtained in [ACF07]. Applications of this point interaction were discussed recently in [HC06], and it is worth mentioning that it appears also in the theory of regular tree graphs [So04]. On the other hand, in distinction to [ACF07] we have here a wider class which contains, in particular, the standard δ interaction of coupling strength $\hat\lambda$ [AGHH05] corresponding to $c_1 = 1$ and $c_2 = 0$. Spectral and scattering properties of $\overline{H}^r$ are well known [EG99] and we recall them only briefly:

Proposition 2. *For any $-\infty < \hat\lambda \leqslant \infty$, the essential spectrum of $\overline{H}^r$ is absolutely continuous and coincides with the interval $[0,\infty)$. Furthermore, for $\hat\lambda > 0$ there are no eigenvalues, while for $\hat\lambda < 0$ there is just one negative eigenvalue given by $k^2 = k_0^2 = -\frac{1}{4}\hat\lambda^2(c_1^2+c_2^2)^{-1}$ and the corresponding normalized eigenfunction is*

$$\psi_0 = \sqrt{\frac{|\hat\lambda|}{2}}\frac{1}{c_1^2+c_2^2}\left\{\begin{matrix} (c_1-c_2)e^{ik_0 s} & s>0 \\ (c_1+c_2)e^{-ik_0 s} & s<0 \end{matrix}\right\}, \qquad k_0 = \frac{i|\hat\lambda|}{2(c_1^2+c_2^2)}\,, \ \ \hat\lambda < 0\,.$$

Finally, for $\hat\lambda = 0$ the operator $\overline{H}^r$ has a zero energy resonance. The on-shell scattering matrix at energy k^2, $k \geqslant 0$ is given by $\mathscr{S}(k) = \begin{bmatrix} \mathscr{T}^l(k) & \mathscr{R}^r(k) \\ \mathscr{R}^l(k) & \mathscr{T}^r(k)\end{bmatrix}$ with the amplitudes

$$\mathscr{T}^{\{l,r\}}(k) = \frac{2k(c_1^2-c_2^2)}{2k(c_1^2+c_2^2)+i\hat\lambda}\,, \qquad \mathscr{R}^{\{l,r\}}(k) = \pm\frac{4kc_1c_2 \mp i\hat\lambda}{2k(c_1^2+c_2^2)+i\hat\lambda}\,.$$

Let now G_k be the resolvent of the free Laplacian on $\mathbb{R}$, it is a bounded operator-valued analytical function of k^2 for $k^2 \in \mathbb{C}\backslash\mathbb{R}^+$ and $\operatorname{Im} k > 0$, with the integral kernel given by

$$G_k(s-s') = \frac{i}{2k}e^{ik|s-s'|} \qquad k^2 \in \mathbb{C}\backslash\mathbb{R}^+,\ \operatorname{Im} k > 0\,.$$

By Krein's formula [ABD95, EG99] the integral kernel of the resolvent $\overline{R}^d(k^2) := (\overline{H}^d - k^2)^{-1}$ is

$$\overline{R}^d(k^2, s, s') = G_k(s-s') + 2ikG_k(s)G_k(s')\,, \quad k^2 \in \mathbb{C}\backslash\mathbb{R}^+\,,\ \operatorname{Im} k > 0\,,$$

while the integral kernel of the resolvent $\overline{R}^r(k^2) := (\overline{H}^r - k^2)^{-1}$ equals

$$\begin{aligned}\overline{R}^r(k^2; s, s') =& G_k(s-s') + 2ik\frac{2kc_2^2 + i\hat\lambda}{2k(c_1^2+c_2^2)+i\hat\lambda}G_k(s)G_k(s') + \frac{4ic_2^2}{2k(c_1^2+c_2^2)+i\hat\lambda}G_k'(s)G_k'(s') \\ &+ \frac{4kc_1c_2}{2k(c_1^2+c_2^2)+i\hat\lambda}\left[G_k(s)G_k'(s') + G_k'(s)G_k(s')\right]\,, \quad k^2 \in \rho(\overline{H}^r)\,,\ \operatorname{Im} k > 0\,.\end{aligned}$$

Our main result can be now stated as follows:

Theorem 2.1. *Suppose that for every $0 < \varepsilon \leqslant \varepsilon_0$ the curve C_ε has no self-intersections, γ is piecewise C^2 with a compact support, and γ', γ'' are bounded. Assuming $\alpha > 5/2$, we have:*

(i) If $-\frac{d^2}{ds^2} - \frac{1}{4}\gamma^2(s)$ does not have a zero energy resonance, then

$$\mathrm{u}-\lim_{\varepsilon\to 0} \overline{R}^{\varepsilon}_{n,m}(k^2) = \delta_{n,m}\overline{R}^{d}(k^2) \qquad k^2 \in \mathbb{C}\backslash\mathbb{R},\ \ \mathrm{Im}\ k > 0\,.$$

(ii) If, on the other hand, $-\frac{d^2}{ds^2} - \frac{1}{4}\gamma^2(s)$ has a zero energy resonance, then

$$\mathrm{u}-\lim_{\varepsilon\to 0} \overline{R}^{\varepsilon}_{n,m}(k^2) = \delta_{n,m}\overline{R}^{r}(k^2) \qquad k^2 \in \mathbb{C}\backslash\mathbb{R},\ \ \mathrm{Im}\ k > 0\,,$$

where the constants c_1, c_2 and $\hat{\lambda}$, defined in (2.5) and (2.6), are obtained by setting $\overline{V} = -\frac{1}{4}\gamma^2$ and $\delta_{n,m}$ indicates the Kronecker symbol, $\delta_{n,m} = 0$ if $n \neq m$ and $\delta_{n,n} = 1$.

3 The limit of short range potentials in dimension one

The main ingredient in the proof of Theorem 2.1 is the analysis of scaling properties of one dimensional Hamiltonians. Specifically, we will find the limiting behaviour as $\varepsilon \to 0$ for

$$\overline{H}_\varepsilon := -\frac{d^2}{ds^2} + \frac{\lambda(\varepsilon)}{\varepsilon^2}\overline{V}\left(\frac{s}{\varepsilon}\right), \qquad s \in \mathbb{R}\,.$$

Recall that this problem is well studied if the limit is considered, roughly speaking, around the free operator [AGHH05]. The case which involves threshold resonances is different and in a sense similar to approximations of point interactions in dimension two and three mentioned above. It is useful to discuss this issue separately because in our opinion it is of independent interest as approximation of a class of point perturbations of the Laplacian in dimension one with scaled potentials. Let us stress that the δ'-type interactions do not belong to this class – a way to approximate them by regular potentials can be found in [ENZ01]. The main idea of our analysis comes from the work [BGW85].

In the following we suppose that the conditions (2.4) are satisfied. As $\lambda(\varepsilon)$ is real analytic near the origin by assumption we can make the expansion (2.2) for small ε more specific writing

$$\lambda(\varepsilon) = 1 + \sum_{n=1}^{\infty} \lambda_n \varepsilon^n\,. \tag{3.1}$$

For every $\varepsilon > 0$ the resolvent of $\overline{H}_\varepsilon$ is a bounded operator-valued analytical function of k^2 as long as $k^2 \in \mathbb{C}\backslash\mathbb{R}^+$, $k^2 \notin \sigma_p(\overline{H}_\varepsilon)$ and Im $k > 0$, where $\sigma_p(\overline{H}_\varepsilon)$ denotes the point spectrum of $\overline{H}_\varepsilon$. As usual we factorize the interaction using the functions

$$v(s) := |\overline{V}(s)|^{1/2}\,, \quad u(s) := \operatorname{sgn}[\overline{V}(s)]|\overline{V}(s)|^{1/2}\,,$$

which allows us to write $(\overline{H}_\varepsilon - k^2)^{-1}$ as in [AGHH05], namely

$$(\overline{H}_\varepsilon - k^2)^{-1} = G_k - \frac{\lambda(\varepsilon)}{\varepsilon} A_\varepsilon(k) T_\varepsilon(k) C_\varepsilon(k)\,, \tag{3.2}$$

where

$$T_\varepsilon(k) = \left[1 + \lambda(\varepsilon) u G_{\varepsilon k} v\right]^{-1} \qquad \mathrm{Im}\ k \geqslant 0,\ k \neq 0,\ k^2 \notin \sigma_p(\overline{H_\varepsilon})$$

and $A_\varepsilon(k)$, $C_\varepsilon(k)$ are defined via their integral kernels, $A_\varepsilon(k; s, s') = G_k(s - \varepsilon s')v(s')$ and $C_\varepsilon(k; s, s') = u(s)G_k(\varepsilon s - s')$, respectively. We are interested in the behaviour of $T_\varepsilon(k)$ as $\varepsilon \to 0$. To this aim we define the operators P and Q by

$$P := \frac{1}{(v,u)}(v, \cdot\,)u\,, \quad Q := 1 - P$$

where $(\cdot\,,\cdot)$ denotes the standard scalar product in $L^2(\mathbb{R})$; let us notice that by assumption we have $(v, u) = \int_{\mathbb{R}} \overline{V}(s)ds \neq 0$. The operator $T_\varepsilon(k)$ can be written as in [BGW85],

$$T_\varepsilon(k) = \left[1 + \frac{i(v,u)}{2\varepsilon k} P + \widetilde{M}_\varepsilon(k)\right]^{-1} \tag{3.3}$$

where $\widetilde{M}_\varepsilon(k) \in \mathscr{B}(L^2, L^2)$, the Banach space of bounded operators from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, for every $\varepsilon > 0$ and Im $k > 0$. Furthermore, if $e^{a|\cdot|}\overline{V} \in L^1(\mathbb{R})$ holds for some $a > 0$ then $\widetilde{M}_\varepsilon$ is analytic with respect to ε for $\varepsilon > -a/(2\,\mathrm{Im}\ k)$ and the following series expansion converges in the $\mathscr{B}(L^2, L^2)$-norm,

$$\widetilde{M}_\varepsilon(k) = \sum_{n=0}^{\infty} \varepsilon^n \widetilde{m}_n(k)\,,$$

where

$$\widetilde{m}_n(k) := (ik)^n m_n + \frac{i\lambda_{n+1}(v,u)}{2k} P + \sum_{j=0}^{n} \lambda_{n-j}(ik)^j m_j \qquad n = 0, 1, 2, \ldots\,.$$

The operators m_n are Hilbert-Schmidt and do not depend on k, their integral kernels being given by the expressions

$$m_n(s, s') = -u(s)\, \frac{|s - s'|^{n+1}}{2(n+1)!}\, v(s')\,.$$

The behaviour of $T_\varepsilon(k)$ as $\varepsilon \to 0$ depends strongly on the presence of a zero energy resonance for the Hamiltonian $\overline{H}$. Under the assumptions (2.4) the presence of such a resonance is equivalent to the existence of a function $\varphi_0 \in L^2(\mathbb{R})$ which satisfies the relation

$$\varphi_0 + QM_0Q\varphi_0 = 0\,. \tag{3.4}$$

Furthermore, if such a φ_0 exists, it can be chosen real, in which case the constants c_1, c_2 and $\hat{\lambda}$ defined in (2.5) and (2.6), respectively, are related to φ_0 by

$$c_1 = \frac{(v, m_0\varphi_0)}{(v,u)}\,, \quad c_2 = \frac{1}{2}((\cdot)v, \varphi_0)\,, \quad \hat{\lambda} = \lambda_1(\tilde{\varphi}_0, \varphi_0)\,.$$

with $\tilde{\varphi}_0(s) := \operatorname{sgn}[\overline{V}(s)]\varphi_0$, and $u(s)\psi_r(s) = -\varphi_0(s)$ holds a.e. – cf. Lemma 2.2. in [BGW85]. Let us introduce the operator

$$P_0 := \begin{cases} 0 & \text{in the case I} \\ \dfrac{(\tilde{\varphi}_0, \cdot)\varphi_0}{(\tilde{\varphi}_0, \varphi_0)} & \text{in the case II} \end{cases}$$

and the complementary projection $Q_0 := 1 - P_0$. From Lemma 3.1 in [BGW85] we infer that for $\varepsilon \in \mathbb{C}\backslash\{0\}$ small enough the following norm convergent series expansion holds,

$$[1 + Qm_0Q + \varepsilon]^{-1} = \frac{P_0}{\varepsilon} + \sum_{n=0}^{\infty} (-\varepsilon)^n T_{red}^{n+1}\,,$$

where $T_{red} = \mathrm{u} - \lim_{\varepsilon\to 0}[1 + Qm_0Q + \varepsilon]^{-1}Q_0$ is the reduced resolvent. The following claim is a generalization of Theorem 3.1 in [BGW85]

Lemma 3.1. *Suppose that $\overline{V}$ satisfies the conditions* (2.4) *and take $\lambda(\varepsilon)$ real analytic near the origin and with the series expansion* (3.1). *Assume that $k^2 \notin \sigma_p(\overline{H}_\varepsilon)$, $\operatorname{Im} k > 0$ and additionally, that in the case II $k \neq -i\hat{\lambda}/(2(c_1^2 + c_2^2))$. Then for all ε small enough the operator $T_\varepsilon(k)$ has the following norm-convergent series expansions*

$$T_\varepsilon(k) = \sum_{n=p}^{\infty} \varepsilon^n t_n(k)\,, \tag{3.5}$$

where $p = 0$ in the case I and $p = 1$ in the case II. Moreover, we have

(i) In the case I

$$(v, t_0u) = 0\,;\ \big((\cdot)v, t_0u\big) = \big(v, t_0(\cdot)u\big) = 0\,; \tag{3.6}$$

$$(v, t_1u) = -2ik\,. \tag{3.7}$$

(ii) In the case II

$$t_{-1}u = t^*_{-1}v = 0\,;\ \big((\cdot)v, t_{-1}(\cdot)u\big) = -\frac{4ic_2^2}{2k(c_1^2+c_2^2)+i\hat{\lambda}}\,; \tag{3.8}$$

$$(v, t_0u) = 0\,;\ \big((\cdot)v, t_0u\big) = \big(v, t_0(\cdot)u\big) = \frac{4kc_1c_2}{2k(c_1^2+c_2^2)+i\hat{\lambda}}\,; \tag{3.9}$$

$$(v, t_1u) = -2ik\frac{2kc_2^2+i\hat{\lambda}}{2k(c_1^2+c_2^2)+i\hat{\lambda}}\,. \tag{3.10}$$

Proof. We prove the lemma first in the case II. Let us assume that the equation (3.4) is solved by $\varphi_0 \in L^2(\mathbb{R})$. By using the relation [BGW85]

$$1 + \frac{i(v,u)}{2\varepsilon k}P = \left[Q + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)}P\right]^{-1}$$

in formula (3.3) we obtain

$$T_\varepsilon = \left[1 + Q\widetilde{M}_\varepsilon + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)}P\widetilde{M}_\varepsilon\right]^{-1}\left[Q + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)}P\right]. \tag{3.11}$$

Since $Q\widetilde{m}_0 = Qm_0$ the following norm convergent series expansion holds [BGW85]

$$\big(1 + Q\widetilde{m}_0 + \delta\big)^{-1} = \left[\frac{P_0}{\delta} + \sum_{n=0}^{\infty}(-\delta)^n T_{red}^{n+1}\right]\left[1 - \frac{Qm_0P}{1+\delta}\right]. \tag{3.12}$$

Taking into account that $P_0 - P_0Qm_0P = -P_0m_0$ and performing a simple manipulation, we can set $\delta = -2i\varepsilon k/(v,u)$ and use relation (3.12) in formula (3.11) to obtain

$$\begin{aligned} T_\varepsilon = &\left[1 + \left(\frac{(v,u)}{2i\varepsilon k}P_0m_0 + D_\varepsilon\right)\left(\frac{2i\varepsilon k}{(v,u)} + Q\widetilde{M}^{(1)}_\varepsilon + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)}P\widetilde{M}_\varepsilon\right)\right]^{-1} \\ &\times \left[\frac{(v,u)}{2i\varepsilon k}P_0m_0 + D_\varepsilon\right]\left[Q + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)}P\right], \end{aligned}$$

where

$$D_\varepsilon(k) := \frac{2\varepsilon k}{2\varepsilon k + i(v,u)}P_0Qm_0P + \sum_{n=0}^{\infty}\left(\frac{2i\varepsilon k}{(v,u)}\right)^n T_{red}^{n+1}\left[1 - \frac{Qm_0P}{1 - 2i\varepsilon k/(v,u)}\right]$$

and $\widetilde{M}^{(j)}_\varepsilon(k) = \displaystyle\sum_{n=j}^{\infty}\varepsilon^n\widetilde{m}_n(k)$ with $j = 1, 2, \dots$. After some computation we arrive at the following formula for the operator T_ε,

$$T_\varepsilon = \big[1 + P_0\widetilde{B} + E_\varepsilon\big]^{-1}\left[\frac{(v,u)}{2i\varepsilon k}P_0m_0 + D_\varepsilon\right]\left[Q + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)}P\right], \tag{3.13}$$

where

$$\widetilde{B}(k) = m_0 + \frac{(v,u)}{2ik} m_0 Q \widetilde{m}_1(k) - m_0 P \widetilde{m}_0(k)$$

and

$$\begin{aligned} E_\varepsilon(k) =& \frac{(v,u)}{2i\varepsilon k} P_0 m_0 Q \widetilde{M}_\varepsilon^{(2)}(k) + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)} P_0 m_0 P \widetilde{m}_0(k) - \frac{i(v,u)}{2\varepsilon k + i(v,u)} P_0 m_0 P \widetilde{M}_\varepsilon^{(1)}(k) + \\ &+ D_\varepsilon(k) \left(\frac{2i\varepsilon k}{(v,u)} + Q \widetilde{M}_\varepsilon^{(1)}(k) + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)} P \widetilde{M}_\varepsilon(k) \right). \end{aligned}$$

The operator $\widetilde{B}(k)$ does not depend on ε while $D_\varepsilon(k)$ and $E_\varepsilon(k)$ have with respect to the parameter the following norm convergent series expansions

$$D_\varepsilon(k) = \sum_{n=0}^{\infty} \varepsilon^n d_n(k) \,, \quad E_\varepsilon(k) = \sum_{n=1}^{\infty} \varepsilon^n e_n(k) \,. \tag{3.14}$$

Let us notice that

$$\frac{(\tilde{\varphi}_0, \widetilde{B}\varphi_0)}{(\tilde{\varphi}_0, \varphi_0)} = -1 - \frac{(v,u)}{(\tilde{\varphi}_0, \varphi_0)} (c_1^2 + c_2^2 + i\hat{\lambda}/(2k))$$

and $P_0 \widetilde{B} P_0 = (\tilde{\varphi}_0, \widetilde{B}\varphi_0)/(\tilde{\varphi}_0, \varphi_0) P_0$. In a similar way as in [BGW85] we can explicitly evaluate for $k \neq -i\hat{\lambda}/(2(c_1^2 + c_2^2))$ the inverse of $1 + P_0 \widetilde{B}$ obtaining

$$[1 + P_0 \widetilde{B}]^{-1} = 1 + \frac{(\tilde{\varphi}_0, \varphi_0)}{(v,u)} \frac{1}{c_1^2 + c_2^2 + i\hat{\lambda}/(2k)} P_0 \widetilde{B} \,.$$

Formula (3.13) implies that in the case II the norm convergent series expansion (3.5) holds for ε small enough with $p = -1$.

Keeping only the terms corresponding of order ε^{-1} at the right hand side of equation (3.13) and using the relation $P_0 m_0 Q = -P_0$ we obtain

$$t_{-1} = \frac{(v,u)}{2ik} [1 + P_0 \widetilde{B}]^{-1} P_0 m_0 Q = \frac{(\tilde{\varphi}_0, \varphi_0)}{2ik} \frac{1}{(c_1^2 + c_2^2 + i\hat{\lambda}/(2k))} P_0 \,.$$

Relations (3.8) follow from $P_0 u = P_0^* v = 0$ and $((\cdot)v, P_0(\cdot)u) = 4c_2^2/(\tilde{\varphi}_0, \varphi_0)$. Inspecting the terms of order zero in ε at the right hand side of (3.13) we obtain

$$t_0 = [1 + P_0 \widetilde{B}]^{-1} [-P_0 m_0 P + d_0 Q - e_1 t_{-1}] \,.$$

The relation $(v, t_0 u) = 0$ is a consequence of the fact that $\big([1 + P_0 \widetilde{B}]^{-1}\big)^* v = v$. Moreover, by a direct calculation based on the relation $T_{red} P = P T_{red} = P$ similar to [BGW85] one can check that

$$((\cdot)v, t_0 u) = -\big((\cdot)v, [1 + P_0 \widetilde{B}]^{-1} P_0 m_0 u\big)$$

and

$$(v, t_0(\cdot)u) = -(v, e_1 t_{-1}(\cdot)u)$$

from which the relations (3.9) follow. Formula (3.10) is obtained by considering the terms of order ε at the right hand side of equation (3.13) in combination with the relation

$$(v, t_1 u) = -\frac{2ik}{(v,u)}(v, d_0 u) - (v, e_1[1 + P_0\widetilde{B}]^{-1} P_0 m_0 u)\,.$$

It remains to deal with the case I, in such a case $P_0 = 0$ and the equation (3.13) becomes

$$T_\varepsilon = \big[1 + E_\varepsilon\big]^{-1} D_\varepsilon \left[Q + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)} P\right], \tag{3.15}$$

where

$$D_\varepsilon(k) = \sum_{n=0}^{\infty} \left(\frac{2i\varepsilon k}{(v,u)}\right)^n T_{red}^{n+1} \left[1 - \frac{Q m_0 P}{1 - 2i\varepsilon k/(v,u)}\right]$$

and

$$E_\varepsilon(k) = D_\varepsilon(k) \left(\frac{2i\varepsilon k}{(v,u)} + Q\widetilde{M}_\varepsilon^{(1)}(k) + \frac{2\varepsilon k}{2\varepsilon k + i(v,u)} P \widetilde{M}_\varepsilon(k)\right).$$

The series expansions (3.14) still hold, and the norm convergent series expansion (3.5) in case I is valid with $p = 0$. Let us notice that $[1 + E_\varepsilon]$ is invertible for $\varepsilon \geqslant 0$ with ε small enough, and consequently, it is not necessary to assume $k \neq -i\hat{\lambda}/(2(c_1^2 + c_2^2))$. In the case I we thus have

$$t_0 = d_0 Q = T_{red}[1 - Q m_0 P] Q\,,$$

from which it easily follows that $(v, t_0 u) = 0$, and from $P T_{red} = T_{red} P = P$ one gets relations (3.6). The terms of order ε at the right hand side of equation (3.15) give

$$t_1 = \frac{2k}{i(v,u)} d_0 P + d_1 Q - e_1 d_0 Q\,,$$

the formula (3.7) then follows from $(v, d_0 u) = (v, u)$. $\square$

With this result at hand we can follow the argument line of [ACF07] to establish the norm resolvent convergence of the Hamiltonian $\overline{H}_\varepsilon$ to $\overline{H}^d$ or $\overline{H}^r$, depending on the potential $\overline{V}$; we omit the details. Using formulae (3.6)–(3.10) in the proof of Lemma 1 of [ACF07] we arrive at the following conclusion:

Theorem 3.1. *Suppose that $\overline{V}$ satisfies the conditions* (2.4) *and $\lambda(\varepsilon)$ is real analytic near the origin having the series expansion* (3.1)*. Then we have*

(i) In the case I

$$\mathrm{u} - \lim_{\varepsilon \to 0} (\overline{H}_\varepsilon - k^2)^{-1} = \overline{R}^d(k^2) \qquad k^2 \in \mathbb{C}\backslash\mathbb{R},\ \mathrm{Im}\ k > 0\,.$$

(ii) In the case II

$$\mathrm{u} - \lim_{\varepsilon \to 0} (\overline{H}_\varepsilon - k^2)^{-1} = \overline{R}^r(k^2) \qquad k^2 \in \mathbb{C}\backslash\mathbb{R},\ \mathrm{Im}\ k > 0\,.$$

4 Proof of Theorem 2.1

Also the rest of the proof of the main result now follows closely [ACF07] so it is sufficient to sketch the argument. It splits into two steps. The first was dealt with in the previous section, the second step consists of the proof of the claim given below. Since the latter is essentially as Lemma 3 in [ACF07], we just state it omitting the details.

Lemma 4.1. *Suppose that C_ε has no self-intersections for every $0 < \varepsilon \leqslant \varepsilon_0$, and moreover, that γ is piecewise C^2, has compact support. and γ', γ'' are bounded. Fix an $\alpha > 5/2$ and define H_ε^γ as the closure of the e.s.a. operator*

$$H_{0\varepsilon}^\gamma := -\frac{\partial^2}{\partial s^2} - \frac{1}{\varepsilon^{2\alpha}}\frac{\partial^2}{\partial u^2} - \frac{\lambda(\varepsilon)}{\varepsilon^2}\frac{\gamma(s/\varepsilon)^2}{4}$$

with the domain

$$\mathscr{D}(H_{0\varepsilon}^\gamma) := \{\psi \in L^2(\Omega') |\, \psi \in C^\infty(\Omega')\,,\, \psi(s,d) = \psi(s,-d) = 0\,,\, H_{0\varepsilon}^\gamma \psi \in L^2(\Omega')\}\,.$$

Defining the matrix elements $R_{n,m}^{\gamma,\varepsilon}$ with respect to the transverse modes ϕ_n and ϕ_m,

$$R_{n,m}^{\gamma,\varepsilon}(k^2; s, s') = \int_{-d}^{d} du\, du' \phi_n(u)(H_\varepsilon^\gamma - k^2 - E_{\varepsilon,m})^{-1}(s,u,s',u')\phi_m(u')\,,$$

we have

$$\mathrm{u} - \lim_{\varepsilon\to 0}\left(R_{n,m}^{\varepsilon}(k^2) - R_{n,m}^{\gamma,\varepsilon}(k^2)\right) = 0 \qquad k^2 \in \mathbb{C}\backslash\mathbb{R},\ \mathrm{Im}\ k > 0\,.$$

In analogy with [ACF07] Theorem 2.1 is now obtained by combination of Theorem 3.1 and Lemma 4.1.

Acknowledgment: The research was partially supported by GAAS and MEYS of the Czech Republic under projects A100480501 and LC06002, by the Collaborative Research Center (SFB) 611 "Singular Phenomena and Scaling in Mathematical Models" and by the Deutsche Akademische Austauschdienst (DAAD).

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